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A note on the dynamic boundary conditions in Munk-like circulation models

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Abstract. The present note deals with Munk's ocean model and proposes an alternative approach to find its solution, with special regard to the western boundary layer. We introduce a suitable "distance" between the related Sverdrup streamfunction and all the admissible streamfunctions which are valid in the western boundary layer. We prove that such distance has a minimum that singles out a unique solution. Unlike the traditional method, this procedure works without assuming a priori any dynamic boundary condition.

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1 Introduction

The wind-driven ocean circulation theory is, since the beginning of last century, one of the keystones of Physical Oceanography. In the course of its evolution, it has been able to give more and more detailed answers to general and fundamental problems, such as the propagation of the large scale fluid motion into the depths of the oceans, the formation of westward intensified currents in all the major oceans of the Earth, the dynamics of the equatorial countercurrents, the structure of the planetary thermocline and many others. The core of the theory is the Sverdrup balance that shows how the oceanic transport reacts to the vorticity put in by the wind but it does not give any information on the vorticity erosion that takes place in the body of the water. In problems dealing with oceanic basins as a whole or their western regions, another ingredient is therefore necessary to complete the dynamical picture of the system, the vorticity dissipation. In the quasigeostrophic framework, which is widely used in a large class of wind-driven ocean circulation models, the dissipation is mostly realized through a sink of relative vorticity (bottom friction) or the lateral diffusion of the same quantity. The lateral diffusion of relative vorticity poses a special problem. In fact, because of its analytical form, the diffusion term raises the order of the vorticity differential equation and thus the specification of additive, so called dynamic, boundary conditions (dbcs) is demanded to single out a unique solution from the model. Actually, several dbcs are mathematically admissible but the existence and the uniqueness of the "true" one (if any) is an open question. This conceptual difficulty is well known since Munk [1] and it is quoted also recently (for instance Kamenkovich *et al.* [2], Pedlosky [3] and McWilliams [4]). In the literature, the indeterminateness of these dbcs is systematically circumvented by imposing arbitrarily chosen conditions, for instance no-slip, free-slip, partial-slip, super slip, which, however, are not explained in a deductive way.

In the present note we take into account, for simplicity, Munk's ocean model and propose an alternative approach to finding its solution under the assumption that it admits a western boundary layer. The starting point relies on the fact that the Sverdrup solution ψ_I, because of its typical length scale which is comparable with that of the basin itself, tends to fill the whole fluid domain thus squeezing the western boundary layer towards the westernmost part of the basin. In spite of this, it is well known that the western boundary layer solution ψ_W is "far" from that of Sverdrup so one can evaluate a "distance" between ψ_W and ψ_I in a suitable function space. Keeping fixed ψ_I, we prove that for varying ψ_W characterized by an arbitrary function $C(y)$, this distance depends on $C(y)$. The minimum of the distance determinates a unique dbc and eliminates the indeterminateness due to $C(y)$. Conversely, one can use this same dbc to find ψ_W through $C(y)$.

2 Review of Munk's model

The framework of our investigation is Munk's model of the basin-scale ocean circulation for a homogeneous ocean, governed by the nondimensional vorticity equation in the form

$$\frac{\partial \psi}{\partial x} = T(x,y) + (\delta_M/L)^3\, \nabla^4 \psi, \tag{1}$$

where $T(x,y) \equiv \hat{k} \cdot \vec{\nabla} \times \vec{\tau}$ is the wind forcing and δ_M/L is the nondimensional boundary layer thickness.

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Equation (1) is referred to the fluid domain

$$D = [-\pi/2 \le x \le \pi/2] \times [-\pi/2 \le y \le \pi/2] \quad (2)$$

of the beta plane and the no mass flux across the boundary of D, *i.e.*

$$\psi(x, \pm\pi/2) = 0 \ \forall x \in [-\pi/2, \pi/2], \\ \psi(\pm\pi/2, y) = 0 \ \forall y \in [-\pi/2, \pi/2], \quad (3)$$

is prescribed for the flow. The latitudes $y = \pm\pi/2$ select two consecutive circles where the forcing $\hat{k}\cdot\vec{\nabla}\times\vec{\tau}$ vanishes, *i.e.*

$$T(x, \pm\pi/2) = 0 \quad (4)$$

so its form should be chosen to be consistent with this condition. The related Sverdrup streamfunction ψ_I is the solution of problem

$$\begin{cases} \dfrac{\partial\psi_I}{\partial x} = T(x,y) \\ \psi_I(\pi/2, y) = 0 \end{cases}, \quad (5)$$

that is

$$\psi_I(x,y) = -\int_x^{\pi/2} T(\lambda, y)\,\mathrm{d}\lambda. \quad (6)$$

It is well known that streamfunction (6) is a local solution of problem (1), (3) if $O(\delta_M/L) \ll 1$. Because of (4) and (6), the Sverdrup streamfunction satisfies conditions (3) along the zonal and eastern walls of D and it represents, aside from the western boundary layer, the solution of the circulation problem provided that we assume also free-slip dbcs along these walls. This is a direct consequence of the identity

$$\nabla^2\psi_I = -\psi_I$$

that easily comes from (6).

Hereafter we understand free-slip dbcs along the zonal and the eastern boundaries and focus our attention only to the western region ($-\pi/2 \le x < \delta_M/L$) where we attach the stretched coordinate ξ defined by the equation

$$\delta_M\xi = L(x + \pi/2) \quad (7)$$

and express the local solution ψ_W as

$$\psi_W = \psi_I(x,y) + \phi(\xi, y). \quad (8)$$

In (8) $\phi(\xi, y)$ is the well known matching term of the form [3]

$$\phi(\xi, y) \equiv \exp(-\xi/2)\left[C(y)\sin\left(\sqrt{3}\xi/2\right) + I(y)\cos\left(\sqrt{3}\xi/2\right)\right]. \quad (9)$$

In (9),

$$I(y) = \int_{-\pi/2}^{\pi/2} T(\lambda, y)\,\mathrm{d}\lambda, \quad (10)$$

while, usually, the function $C(y)$ is singled out once the dbc is specified in $\xi = 0$ (*i.e.* in $x = -\pi/2$, see (7)). In particular, from (8) and (9) we evaluate

$$\left[\frac{\partial\psi_W}{\partial x}\right]_{\substack{x=-\pi/2 \\ \xi=0}} = T(-\pi/2, y) - \frac{L}{2\delta_M}\left[I(y) - \sqrt{3}C(y)\right] \quad (11)$$

and

$$\left[\frac{\partial^2\psi_W}{\partial x^2}\right]_{\substack{x=-\pi/2 \\ \xi=0}} = \frac{\partial T(-\pi/2, y)}{\partial x} - \frac{1}{2}\left(\frac{L}{\delta_M}\right)^2 \times \left[I(y) + \sqrt{3}C(y)\right]. \quad (12)$$

As $O(L/\delta_M) \gg 1$ while T and $\frac{\partial T}{\partial x}$ are $O(1)$, from (11) we infer that at $x = -\pi/2$,

$$\frac{\partial\psi_W}{\partial x} = 0 \Leftrightarrow C(y) = \frac{1}{\sqrt{3}} I(y) \quad (13)$$

while, from (12),

$$\frac{\partial^2\psi_W}{\partial x^2} = 0 \Leftrightarrow C(y) = -\frac{1}{\sqrt{3}} I(y)\cdot \quad (14)$$

In other words, both no-slip and free-slip dbcs imply a definite proportionality between the functions $C(y)$ and $I(y)$ according to (13) and (14) respectively. Perhaps the no-slip dbc is more usual than that of free-slip; in any case, a detailed description of this last solution can be found, for instance, in [3], Section 2.7.

In view of the following, we recall also the partial-slip dbc which, along the western boundary, takes the general form $\frac{\partial^2\psi_W}{\partial x^2} - \alpha\frac{\partial\psi_W}{\partial x} = 0$, α being a constant to be defined. Assume now the generic relationship

$$C(y) = \sigma\, I(y) \quad \text{where} \quad \sigma \in \Re,\ \sigma \ne \pm 1/\sqrt{3}. \quad (15)$$

If (15) holds, then equations (11) and (12) give respectively, at $x = -\pi/2$,

$$\frac{\partial\psi_W}{\partial x} = -\frac{L}{2\delta_M}\left(1 - \sigma\sqrt{3}\right) I(y), \quad (16)$$

$$\frac{\partial^2\psi_W}{\partial x^2} = -\frac{1}{2}\left(\frac{L}{\delta_M}\right)^2\left(1 + \sigma\sqrt{3}\right) I(y). \quad (17)$$

From (16) and (17) the partial-slip dbc in the form

$$\frac{\partial^2\psi_W}{\partial x^2} = \frac{L}{\delta_M}\frac{1 + \sigma\sqrt{3}}{1 - \sigma\sqrt{3}}\frac{\partial\psi_W}{\partial x}, \quad \sigma \ne \pm 1/\sqrt{3} \quad (18)$$

follows. No-slip and free-slip dbcs imply the uniqueness of the solution of Munk's model. The same is true also for the partial-slip dbc (18) provided that $-1/\sqrt{3} < \sigma < 1/\sqrt{3}$. To explain this point, we start from two hypothetical solutions, say ψ_1 and ψ_2 of problem (1), (3), and consider the equation satisfied by their difference $h(x,y) \equiv \psi_1(x,y) - \psi_2(x,y)$, that is, recalling (1),

$$\frac{\partial h}{\partial x} = (\delta_M/L)^3\,\nabla^4 h. \quad (19)$$

Multiplication of (19) by h and the subsequent integration on D with the aid of (3) yields

$$\oint_{\partial D} \nabla^2 h \vec{\nabla} h \cdot \hat{n}\, \mathrm{d}s = \int_D \left(\nabla^2 h\right)^2 \mathrm{d}x\mathrm{d}y. \tag{20}$$

As the considered dbcs are linear and homogeneous, if ψ_1 and ψ_2 satisfy one of them, also h verifies the same condition. Therefore no-slip and free-slip dbcs imply the vanishing of the line integral at the l.h.s. of (20) so $\int_D (\nabla^2 h)^2 \mathrm{d}x\mathrm{d}y = 0$ and thus $h = 0$. Thus uniqueness immediately follows.

Partial-slip dbc along the western side and free-slip elsewhere imply

$$\oint_{\substack{\text{western}\\ \text{wall}}} \nabla^2 h \vec{\nabla} h \cdot \hat{n}\, \mathrm{d}s = \int_D \left(\nabla^2 h\right)^2 \mathrm{d}s \tag{21}$$

but, along the western wall, $\hat{n}\cdot\hat{i} = -1$ so $\nabla^2 h \vec{\nabla} h \cdot \hat{n}\, \mathrm{d}s = -\frac{\partial^2 h}{\partial x^2}\frac{\partial h}{\partial x}$. Because of (18), $\frac{\partial^2 h}{\partial x^2} = \frac{L}{\delta_M}\frac{1+\sigma\sqrt{3}}{1-\sigma\sqrt{3}}\frac{\partial h}{\partial x}$ and hence $-\int_{\substack{\text{western}\\ \text{wall}}} \frac{\partial^2 h}{\partial x^2}\frac{\partial h}{\partial x}\mathrm{d}s = -\frac{L}{\delta_M}\frac{1+\sigma\sqrt{3}}{1-\sigma\sqrt{3}}\int_{\substack{\text{western}\\ \text{wall}}} \left(\frac{\partial h}{\partial x}\right)^2 \mathrm{d}s$. Thus we have from (21)

$$-\frac{L}{\delta_M}\frac{1+\sigma\sqrt{3}}{1-\sigma\sqrt{3}} \int_{\substack{\text{western}\\ \text{wall}}} \left(\frac{\partial h}{\partial x}\right)^2 \mathrm{d}s = \int_D \left(\nabla^2 h\right)^2 \mathrm{d}s.$$

We conclude that a sufficient condition in order that h be identically vanishing is $\frac{1+\sigma\sqrt{3}}{1-\sigma\sqrt{3}} > 0$ whence the uniqueness of the solution holds if $-1/\sqrt{3} < \sigma < 1/\sqrt{3}$.

3 A "distance" between the western and the interior solutions and the related dynamic boundary condition

Since, once the forcing is fixed, the difference $\psi_W - \psi_I$ given by (9) depends only on $C(y)$ through ϕ, it makes sense to explore how a suitable "distance" of ψ_W from ψ_I, say $d(\psi_W, \psi_I)$ is minimized by $||C|| \equiv \left\{\int_{-\pi/2}^{\pi/2} C^2(y)\mathrm{d}y\right\}^{1/2}$ and how $||C||$ is possibly related to a definite dbc.

To establish $d(\psi_W, \psi_I)$, we consider first the norm

$$||\psi_W - \psi_I||_1 \equiv \left\{\int_{-\pi/2}^{\pi/2}\mathrm{d}y \int_0^{+\infty} \mathrm{d}\xi (\psi_W - \psi_I)^2\right\}^{1/2} \tag{22}$$

which quantifies in a quite obvious way the departure of the local solution (8) from that of the interior (6). With the aid of the further definitions

$$\langle C(y)I(y)\rangle = \int_{-\pi/2}^{\pi/2}\mathrm{d}y\, C(y)I(y),$$
$$||I|| \equiv \left\{\int_{-\pi/2}^{\pi/2} I^2(y)\mathrm{d}y\right\}^{1/2} \tag{23}$$

the substitution of (9) into (22) yields

$$||\psi_W - \psi_I||_1^2 = \frac{3}{8}||C||^2 + \frac{\sqrt{3}}{4}\langle CI\rangle + \frac{5}{8}||I||^2. \tag{24}$$

Second, we consider the norm

$$||\psi_W - \psi_I||_2 \equiv \left\{\int_{-\pi/2}^{\pi/2}\mathrm{d}y \int_0^{+\infty} \mathrm{d}\xi \left[\frac{\partial}{\partial\xi}(\psi_W - \psi_I)\right]^2\right\}^{1/2} \tag{25}$$

which quantifies the departure of the meridional velocity of the western boundary layer from that given by the Sverdrup balance which has the opposite sign. The explicit evaluation of (25) gives, using again (23),

$$||\psi_W - \psi_I||_2^2 = \frac{3}{8}||C||^2 - \frac{\sqrt{3}}{4}\langle CI\rangle + \frac{5}{8}||I||^2. \tag{26}$$

Starting from (24) and (26) we define

$$d(\psi_W, \psi_I) = \left\{||\psi_W - \psi_I||_I^2 + ||\psi_W - \psi_I||_2^2\right\}^{1/2}, \tag{27}$$

that is to say

$$d(\psi_W, \psi_I) = \left\{\frac{3}{4}||C||^2 + \frac{5}{4}||I||^2\right\}^{1/2}. \tag{28}$$

Obviously, a large freedom exists as a defining the distance $d(\psi_W, \psi_I)$, however, for our purposes it is mathematically convenient that $d(\psi_W, \psi_I)$, while depending on $||C||$, does not depend on other functionals of $C(y)$, like $\langle CI\rangle$. Physically, we see from (8, 22, 25) and (27) that $d = \sqrt{2E}$ where E is the sum of the (nondimensional) potential and kinetic energy of the flow in the western boundary layer

$$E = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\mathrm{d}y \int_0^{+\infty}\mathrm{d}\xi \left[\phi^2 + \left(\frac{\partial\phi}{\partial\xi}\right)^2\right].$$

This, under the approximation that, in the considered region, the kinetic energy of the zonal flow can be neglected with respect to that meridional.

By using (27), one can check that $d(\psi_W, \psi_I)$ actually satisfies all the defining axioms of a distance (see the Appendix, concerning with the triangular inequality), *i.e.*:

- $d(\psi_W, \psi_I) \geq 0,\ d(\psi_W, \psi_I) = 0 \Leftrightarrow \psi_W = \psi_I$,
- $d(\psi_W, \psi_I) = d(\psi_I, \psi_W)$,
- $d(\psi_W, \psi_I) \leq d(\psi_W, \varphi) + d(\varphi, \psi_I),\ \forall\varphi$.

From (28) we evaluate the minimum of $d(\psi_W, \psi_I)$ in function of $||C||$ which takes place in

$$||C|| = 0 \tag{29}$$

and is given by

$$d_{\min}(\psi_W, \psi_I) = \frac{\sqrt{5}}{2}||I||. \tag{30}$$

From (10, 23) and (30) the minimum energy E_m in the western boundary layer turns out to be, in function of the wind forcing, $E_m = \frac{5}{8}||I||^2 = \frac{5}{8}\int_{-\pi/2}^{\pi/2} \mathrm{d}y[\int_{-\pi/2}^{\pi/2} \mathrm{d}\lambda\, T(\lambda,y)]^2$. In accordance with this result, using (13) and (14) into (28) we verify that both the no-slip and free-slip conditions imply $E = \frac{3}{4}||I||^2 > E_m$.

More remarkably, equation (29) implies $C(y) \equiv 0$ and therefore (15) holds for $\sigma = 0$. In this way we select from (18) the partial-slip condition

$$\frac{\partial^2 \psi_W}{\partial x^2} - \frac{L}{\delta_M}\frac{\partial \psi_W}{\partial x} = 0 \quad \text{at} \quad x = -\pi/2. \tag{31}$$

From Section 2 we know that boundary condition (31) leads of necessity to a unique solution in which $C(y) \equiv 0$. This fact can be easily verified solving explicitly the problem by resorting to standard boundary layer techniques.

By using (31), we can check that partial-slip dbc along the western side and free-slip elsewhere imply

$$\oint_{\partial D} \nabla^2\psi\, \vec{\nabla}\psi \cdot \hat{n}\, \mathrm{d}s = -\frac{L}{\delta_M} \int_{\substack{\text{western}\\ \text{wall}}} \left(\frac{\partial \psi_W}{\partial x}\right)^2 \mathrm{d}s < 0$$

and therefore these boundary conditions do not add energy to the system [3], Section 2.10.

To explain the structure of the meridional velocity $v = \frac{\partial \psi}{\partial x}$, for different dbcs at the western boundary, we consider the western region $(-\pi/2 \leq x \leq 0)$ of the fluid domain (2) and take the standard forcing $T(y) = -\cos(y)$. Then, if we solve problem (1, 3) and evaluate v at the middle-basin latitude $y = 0$, we find

$$v(x) = -1 + r\exp[-r(x+\pi/2)]\left\{\left(\sqrt{3}\pi - C\right) \times \sin\left\lfloor\sqrt{3}r(x+\pi/2)\right\rfloor + \left(\sqrt{3}C + \pi\right)\cos\left\lfloor\sqrt{3}r(x+\pi/2)\right\rfloor\right\} \tag{32}$$

where $r = \frac{L}{2\delta_M}$. Figure 1 shows the plots of $v(x)$ for

1) $C = -\pi/\sqrt{3} + 1/\left(r\sqrt{3}\right)$ $(= O(\pi/\sqrt{3}))$, *i.e.* in the no-slip case;
2) $C = 0$, *i.e.* in the partial-slip case given by (31);
3) $C = \pi/\sqrt{3}$, *i.e.* in the free-slip case.

Note that, in order to make the comparison meaningful, we have fixed a unique value of the parameter $r(= 4)$. While this yields a physically acceptable profile of $v(x)$ for free-slip dbc, for this same value of r, the boundary layer width in (1) and (2) turns out to be rather large. In any case, nothing prevents to take smaller values of r in these last configurations.

Moreover, we observe in the same figure a sequence of nodes for given values x_n of longitude, which come from the condition $\frac{\partial v}{\partial C}$ applied to (32), that is to say $\mathrm{tg}\lfloor\sqrt{3}r(x_n + \pi/2)\rfloor = \sqrt{3}$.

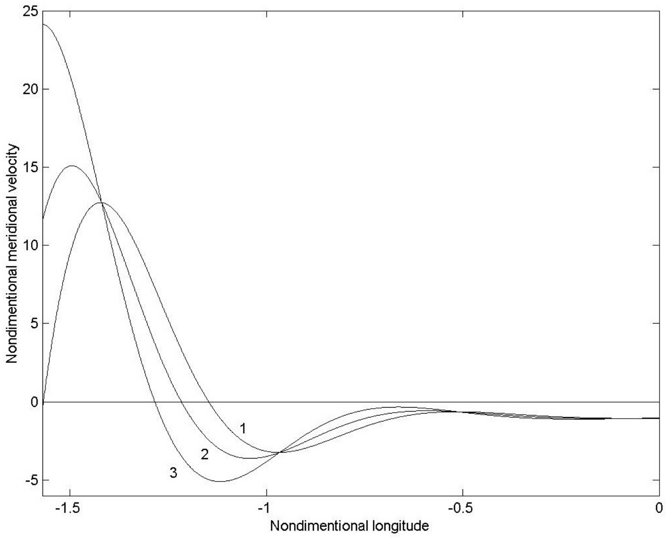

Fig. 1. Profiles of the nondimensional meridional velocity given by (32) for 1) no-slip, 2) partial-slip, 3) free-slip dynamical boundary conditions. The nondimensional longitude is restricted to the half interval $-\pi/2 \leq x \leq 0$. The meridional velocity is referred to the middle-basin latitude.

4 Conclusion

Previous considerations point out the possibility to restate Munk's model by substituting any *a priori* dbc with the request of minimizing the distance (27, 28). The procedure singles out the identically vanishing function $C(y) \equiv 0$ which is equivalent to the use of condition (31) in the standard method. When nonlinear effects are included, another nondimensional parameter, usually denoted by δ_I/L, must be taken into consideration, as well as the ratio δ_I/δ_M. We wish to emphasize the difficulty in generalizing the method outlined in Section 3 in the presence of nonlinearity, although the analytical approach seems to be feasible in the range $0 < \delta_I/\delta_M \ll 1$.

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Appendix

Here we verify the triangular inequality, appearing in the list of the defining axioms of a distance, for d given in (27). We set preliminarily $a_i = ||\psi_W - \varphi||_i$ and $b_i = ||\varphi - \psi_I||_i$. Hence we can write $d(\psi_W, \varphi) = \left(a_1^2 + a_2^2\right)^{1/2}$ and $d(\varphi, \psi_I)(\left(b_1^2 + b_2^2\right)^{1/2}$. We know (see for instance Kolmogorov and Fomin [5]) that

$$||\psi_W - \psi_I||_i \leq a_i + b_i. \tag{A.1}$$

To prove that

$$d(\psi_W, \psi_I) \leq d(\psi_W, \varphi) + d(\varphi, \psi_I),$$

that is to say

$$\left(||\psi_W - \psi_I||_I^2 + ||\psi_W - \psi_I||_2^2\right)^2 \leq \left(a_2^2 + a_2^2\right)^{1/2} + \left(b_1^2 + b_2^2\right)^{1/2}, \quad \text{(A.2)}$$

we bound from above the l.h.s. of (A.2), starting from (A.1), as follows.

$$\begin{aligned}
&\left(||\psi_W - \psi_I||_1^2 + ||\psi_W - \psi_I||_2^2\right)^{1/2} \\
&\leq \left[\left(a_1 + b_1\right)^2 + \left(a_2 + b_2\right)^2\right]^{1/2} \\
&= \left[a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\left(a_1 b_1 + a_2 b_2\right)\right]^{1/2} \\
&\leq \left[a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\left(a_1^2 + a_2^2\right)^{1/2}\left(b_1^2 + b_2^2\right)^{1/2}\right] \\
&= \left(\left(a_1^2 + a_2^2\right)^{1/2} + \left(b_1^2 + b_2^2\right)\right)^{1/2}. \quad \text{(A.3)}
\end{aligned}$$

Last term of (A.3) coincides with the r.h.s. of (A.2), so inequality (A.2) is proved.

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